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A universal Hamilton–Jacobi equation for second-order ODEs

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Abstract. A universal version of the Hamilton–Jacobi equation on $\mathbb{R} \times TM$ arises from the Liouville–Arnol'd theorem for a completely integrable system on a finite-dimensional manifold M. We give necessary and sufficient conditions for such complete integrability to imply a canonical separability of both this universal Hamilton–Jacobi equation and its traditional counterpart. The geodesic case is particularly interesting. We show that these conditions also apply for systems of second-order ordinary differential equations (contact flows) which are not Euler–Lagrange. The Kerr metric, the Toda lattice and a completely integrable contact flow are given as examples.

1. Introduction

The Hamilton–Jacobi equation needs no introduction: it provides a canonical way of formulating and solving Hamiltonian/Lagrangian systems from relativity through classical mechanics to quantum mechanics. The additive separability of this equation in many important examples has been the key to its accessibility and to its ongoing interest for theoreticians. Our paper is concerned with broadening this accessibility by producing an analogue of the equation for arbitrary second-order ordinary differential equations (ODEs; that is, systems which are not necessarily Euler–Lagrange). On the theoretical side we are interested in the relationship between the complete integrability of a system ('half the integrals in involution') and the additive separability of the Hamilton–Jacobi equation. In order to analyse this connection we need to treat the equation from a phase space perspective.

In pursuing these two aims we state at the outset that we do not seek a method for solving the Hamilton–Jacobi equation. Instead we assume complete integrability: these first integrals may have been obtained via Noether's theorem, physical arguments or some other good fortune. Then we ask, *a posteriori*, whether those integrals determine a certain special separation of the Hamilton–Jacobi equation. In effect, we are looking for a 'super-integrabilty' property of a completely integrable system (whose utility we explain later).

The Toda lattice as an interesting example (amongst others). While this system is completely integrable, the corresponding Hamilton–Jacobi equation is not separable in the traditional sense [8, 17] and we show that it is not separable on phase space either. Indeed, the former failure is a result of the latter one.

As we will show there are no obstructions to the formulation of Hamilton–Jacobi equations for arbitrary second-order ordinary differential equations (SODEs). This should be contrasted with the problem of phrasing such a system as a set of Euler–Lagrange equations. In general

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this cannot be done [5, 6], but the Hamilton–Jacobi equation requires only a contact form (which exists in principle for any system of SODEs) and is therefore not limited to Euler–Lagrange systems. Hence this paper is the natural extension of [11] on a similar generalization of the Liouville–Arnol'd theorem to contact flows, and in fact the Hamilton–Jacobi equation appeared during the proof of that theorem.

We proceed as follows.

Our first goal is to formulate our phase space Hamilton–Jacobi equation in the case of Euler–Lagrange systems. This exploits the fact that many geometric features of SODEs are simpler when treated on the evolution space, $\mathbb{R} \times TM$ (M is the configuration space, the evolution space is the Lagrangian equivalent of extended phase space).

We show that the existence of a complete solution of this lifted Hamilton–Jacobi equation is equivalent to complete integrability of the original equations of motion (that is, where we have n first integrals in involution with respect to the contact 2-form).

Then we discuss the separability of the Hamilton–Jacobi equation in the Euler–Lagrange case in terms of coordinates for $\mathbb{R} \times TM$ induced by the involutive integrals via Noether's theorem. Our main result in this second area is a theorem which gives necessary and sufficient conditions on our generalized Hamilton–Jacobi equation to be fully separable in these coordinates.

An important consequence of this type of separability is that the remaining first integrals can be calculated directly without having to solve the Hamilton–Jacobi equation. This is the computational significance of the paper.

We also show that if the usual form of the Hamilton–Jacobi equation is separable, then so is our generalized version. However, the converse is not true unless we add the additional condition that there exist n projectable and commuting symmetries of the Cartan 2-form. This result is the well known case of n ignorable coordinates.

Next we turn to arbitrary systems of SODEs and show that the corresponding Hamilton–Jacobi equation on $\mathbb{R} \times TM$ is always fully separable in the coordinates induced by the (appropriately generalized) Noether's theorem. This result for SODEs is particularly important if the original set of integrals arise from the application of the Noether's theorem or some other source and the conditions of our main theorem are satisfied, we obtain the remaining first integrals without having to solve the Hamilton–Jacobi equation at all.

The guarantee of separability has an immediate consequence for Euler–Lagrange systems, namely that there is a gauge in which the Hamilton–Jacobi equation on $\mathbb{R} \times TM$ is fully separable. We finish with some examples.

While we have attempted to make this paper self-contained, the reader may find reference to our earlier, related paper [11] useful. We will be using standard tangent bundle techniques (see, for example, Crampin and Pirani [3]) and we will use the framework of time-dependent, regular, Lagrangian systems which may be found in Crampin *et al* [4] and Sarlet *et al* [14].

There are a large number of modern treatments of the Hamilton–Jacobi equation; so far as we are aware they are all based on the Hamiltonian formulation of the calculus of variations. We mention the work of Woodhouse [18, 19] because it provided us with an understanding of the separability problem, but more particularly because in [19] there is a tantalizing argument showing that even the most rudimentary separability in the base manifold coordinates guarantees the existence of linear and/or quadratic first integrals for quadratic Hamiltonians. Our results indicate that this is not the case when the separable coordinates are not coordinates for the base manifold, however an appropriate example is not yet available. We also mention Marmo *et al* [9] which has a strongly geometric flavour and a large reference section. However, we emphasize that our paper is concerned with an evolution space treatment of the Hamilton–Jacobi equation, the separability of its solutions and their implication for the separability of the traditional Hamilton–Jacobi equation.

2. Some preliminaries

We suppose that M is some smooth, n-dimensional, second countable, Hausdorff manifold with generic local coordinates (x^a). The *evolution space* is defined as $E := \mathbb{R} \times TM$, with the projection onto the first factor being denoted by $t : E \to \mathbb{R}$. Since all our results are local, all closed forms will be assumed exact. We will regard E as a vector bundle $\tau : E \to \mathbb{R} \times M$. E has generic adapted coordinates (t, x^a, u^a) associated with t and (x^a).

A system of second-order differential equations with local expression

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b) \qquad a = 1, \ldots, n$$

is associated with a smooth vector field Γ on E given in the same coordinates by

$$\Gamma := \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial u^a}.$$

The integral curves on E of this vector field are the natural lifts of the solution curves of our original system of equations.

A Γ-*basic form* (or *invariant form* of Cartan [1]) is a form α on E which satisfies any one of three equivalent conditions:

(a) $\alpha(\Gamma) = 0$ and $\Gamma \lrcorner \, \mathrm{d}\alpha = 0$.
(b) $\mathcal{L}_{f\Gamma}\alpha = 0$ for any smooth function f on E.
(c) α is the pullback of a form from the quotient of E by the action of Γ.

The closed, Γ-basic 1-forms are locally just the exterior derivatives of first integrals of Γ.

A *symmetry* of the differential equation is a local one-parameter Lie group action on E which permutes the integral curves of Γ and, by projection, the solution curves on the base M (although this induced action will not, in general, be the flow of a vector field on M). If an action is generated by a vector field X on E then it is a symmetry of Γ if and only if $\mathcal{L}_X\Gamma = \lambda\Gamma$ for some $\lambda \in C^\infty(E)$. We define a *transverse field*, $[X]$, to be the equivalence class of symmetries differing from X by the addition of a $C^\infty(E)$-multiple of Γ, that is, by the addition of a trivial symmetry. As an example of selection from the equivalence class, we could choose $\hat{X} \in [X]$ so that $\hat{X}(t) = 0$ by setting $\hat{X} = X - X(t)\Gamma$, with X any element of $[X]$. This results in a one-to-one correspondence between transverse fields on E and vector fields on the quotient of E by the action of Γ (because $\mathcal{L}_{\hat{X}}\Gamma = 0$).

When the symmetry group action on E is induced by one on $\mathbb{R} \times M$, the generator on E is the prolongation, $X^{(1)}$, of the generator X on $\mathbb{R} \times M$ (see, for example, [4]).

When the differential equations are the Euler–Lagrange equations of some smooth, regular Lagrangian function L on E, we have a maximal rank 2-form, the *Cartan 2-form* $\mathrm{d}\theta_L$, which is the exterior derivative of the *Cartan* 1-*form*, θ_L. This has local expression

$$\theta_L = L\,\mathrm{d}t + \frac{\partial L}{\partial u^a}(\mathrm{d}x^a - u^a\,\mathrm{d}t).$$

θ_L is just the pullback from $\mathbb{R} \times T^*M$ to E of the Hamilton–Poincaré form, $p_a\,\mathrm{d}q^a - H\,\mathrm{d}t$, by the Legendre transformation. In this case Γ is known as the *Euler field* of the Lagrangian and it is completely determined by the conditions

$$\Gamma \lrcorner \, \mathrm{d}\theta_L = 0 \qquad \Gamma(t) = 1.$$

The first of these expressions are the Euler–Lagrange equations and the second ensures that Γ is that member of the kernel of $\mathrm{d}\theta_L$ whose integral curves are parametrized by the values of the time coordinate.

On the other hand, the necessary and sufficient conditions for a given second-order differential equation field, Γ, to be Euler–Lagrange are called the *Helmholtz conditions* [5, 6, 11].

Now we turn to a discussion of symmetries in the Euler–Lagrange case. Suppose that we have a regular, non-zero Lagrangian on E. (Given a regular Lagrangian, bounded below on E, we can always add a constant so as to make it non-zero.) It is shown in proposition 1 of [11] that symmetries of $\mathrm{d}\theta_L$ are also symmetries of Γ (but the converse is not true). The generators of such symmetries satisfy $\mathcal{L}_X\,\mathrm{d}\theta_L = 0$ as do all other elements of their transverse field. It will be important for us to consider those elements X with $\theta_L(X) = 0$ (proposition 1 of [11] guarantees the existence of this element). Symmetries of $\mathrm{d}\theta_L$ are called *Cartan* symmetries and when they are generated by prolongations of vector fields on $\mathbb{R} \times M$ they are called *Noether* symmetries. Because L is regular, $\mathrm{d}\theta_L$ has maximal rank and the 2-form $\mathrm{d}\theta_L$ provides a bijection $X \mapsto X\lrcorner\,\mathrm{d}\theta_L$ between those vector fields on E satisfying $\mathcal{L}_X\,\mathrm{d}\theta_L = 0$, $\theta_L(X) = 0$ and closed, Γ-basic 1-forms. This well known result is a special case of the Noether–Cartan theorem (see [11] for a proof).

Theorem 2.1. *(Noether–Cartan) The map* $[X] \mapsto X\lrcorner\,\mathrm{d}\theta_L$ *is a bijection of symmetries of* $\mathrm{d}\theta_L$ *(that is,* $\mathcal{L}_X\,\mathrm{d}\theta_L = 0$*) to closed basic 1-forms for* Γ.

We will restrict ourselves to the bijection $\Theta_L : X \mapsto X\lrcorner\,\mathrm{d}\theta_L$ between symmetries of $\mathrm{d}\theta_L$ with $\theta_L(X) = 0$ and closed Γ-basic 1-forms. We can use this map to define involutive first integrals in a natural way which corresponds to the Poisson bracket involution of Hamiltonian mechanics: two smooth first integrals f, g on E with $\mathrm{d}f \wedge \mathrm{d}g \neq 0$ on E are said to be *in involution with respect to* L if $\mathrm{d}\theta_L(X_f, X_g) = 0$. Here $X_f := \Theta_L^{-1}(\mathrm{d}f)$ and $X_g := \Theta_L^{-1}(\mathrm{d}g)$, that is, for example, $\mathcal{L}_{X_f}\,\mathrm{d}\theta_L = 0$, $\theta_L(X_f) = 0$ and $\mathrm{d}f = X_f\lrcorner\,\mathrm{d}\theta_L$. We remark that the map $(f, g) \mapsto -\mathrm{d}\theta_L(X_f, X_g)$ does not turn E into a Poisson manifold as it is only well defined on first integrals. However, it does turn the quotient of E by Γ into a symplectic manifold.

The important properties of involutive first integrals are given in the following propositions from [11]:

Proposition 2.2. *The following conditions are equivalent:*

(a) f *and* g *are involutive first integrals with respect to* L,
(b) $X_f(g) = 0 = X_g(f)$,
(c) $[X_f, X_g] = 0$.

This result shows that each such involutive pair produces a Lagrangian submanifold of $\mathrm{d}\theta_L$ of dimension 3 whose tangent spaces are spanned by X_f, Y_g and Γ.

Proposition 2.3. *If* f *and* g *are involutive first integrals and if* $Y_f \in [X_f]$ *and* $Y_g \in [X_g]$ *are defined by*

$$Y_f := X_f - X_f(t)\Gamma \qquad \text{and} \qquad Y_g := X_g - X_g(t)\Gamma$$

then

$$[Y_f, Y_g] = 0.$$

We can now define a Lagrangian system to be *completely integrable with respect to* L if there exist n first integrals f^a with $\mathrm{d}f^1 \wedge \ldots \wedge \mathrm{d}f^n \neq 0$ which are in involution with respect to L.

3. The Hamilton–Jacobi equation

To define the Hamilton–Jacobi equation from the Lagrangian point of view, suppose that we have a Lagrangian L on E, with Cartan form θ_L. If we consider E as a vector bundle over $\mathbb{R} \times M$ then a section σ of $E \to \mathbb{R} \times M$ defines a Lagrangian submanifold of $\mathrm{d}\theta_L$ (that is, its image is a Lagrangian submanifold of $\mathrm{d}\theta_L$) if

$$\sigma^* \,\mathrm{d}\theta_L = 0.$$

If σ defines a Lagrangian submanifold then locally there is a function S on $\mathbb{R} \times M$ such that

$$\sigma^* \theta_L = \mathrm{d}S.$$

Putting this in coordinates (t, x^a, u^a) we find that

$$\frac{\partial S}{\partial x^a} = \sigma^*(\partial L/\partial u^a) = \sigma^* p_a$$

where p_a is the ath momentum function (but considered as a function on E) and

$$\frac{\partial S}{\partial t} = \sigma^*(L - u^a \partial L/\partial u^a) = -\sigma^* H$$

where H is the Hamiltonian (but considered as a function on E).

These two equations show that S is a solution of the Hamilton–Jacobi equation. Thus every section of $E \to \mathbb{R} \times M$ which defines a Lagrangian submanifold determines a (local) solution of the Hamilton–Jacobi equation.

Conversely, let S be a function on $\mathbb{R} \times M$, and $\tilde{S}$ its total derivative (a function on E):

$$\tilde{S} = \frac{\partial S}{\partial t} + u^a \frac{\partial S}{\partial x^a}.$$

Consider the function $L - \tilde{S}$, and the subset of E on which

$$\frac{\partial}{\partial u^a}(L - \tilde{S}) = 0.$$

This amounts to $\partial S/\partial x^a = \partial L/\partial u^a$, which may be regarded as an implicit equation for u^a in terms of x^a and t. If it is assumed that L is regular then the equation can be solved locally, so determining a local section σ of $E \to \mathbb{R} \times M$. If, in addition, $\partial S/\partial t + \sigma^* H = 0$, then S is a solution of the Hamilton–Jacobi equation. It follows from this that σ defines a Lagrangian submanifold of E with respect to $\mathrm{d}\theta_L$.

Notice that the pair $\partial S/\partial x^a = \partial L/\partial u^a$, $\partial S/\partial t + \sigma^* H = 0$ can also be written as $L - \tilde{S} = 0$, $\frac{\partial}{\partial u^a}(L - \tilde{S}) = 0$. Thus the construction above gives another interpretation of the Hamilton–Jacobi equation: it determines those functions S on E such that the zeros of the function $L - \tilde{S}$ are extrema with respect to variations along the fibres of $E \to \mathbb{R} \times M$.

3.1. Complete solutions of the Hamilton–Jacobi equation

We will now show that a *complete* solution (definition 3.1 to follow) of the Hamilton–Jacobi equation amounts to a smooth family of such sections whose images foliate E, giving rise to a solution on E rather than on $\mathbb{R} \times M$.

To this end, consider n (local) first integrals h^a, not necessarily in involution, with $\mathrm{d}h^1 \wedge \ldots \wedge \mathrm{d}h^n \neq 0$ and set

$$E_c := \{v \in E \,:\, h^a(v) = c^a, a = 1, \ldots, n\}.$$

Let D denote the $(n + 1)$-dimensional integrable distribution whose integral manifolds are the E_c, while σ_c is the section of $\tau : E \to \mathbb{R} \times M$ whose image is E_c. The foliated derivative is just the restriction of d to the distribution D (see [11]).

Definition 3.1. *A pair* $(\{h^1,\dots,h^n\},G)$ *or equivalently* (D,G) *to be a* complete solution *of the Hamilton–Jacobi equation (on E) if it satisfies*

$$\mathrm{d}_D G = \theta_L.$$

This choice of definition is made for the following reasons: a submanifold E_c is a solution of the Hamilton–Jacobi equation if it is Lagrangian for $\mathrm{d}\theta_L$. From proposition 2.2 this is so precisely when the integrals are involutive, that is,

$$\mathrm{d}\theta_L(X_{h^a},X_{h^b})=0 \qquad \Longleftrightarrow \qquad \sigma_c^*\,\mathrm{d}\theta_L=0$$

and so

$$\sigma_c^*\theta_L=\mathrm{d}S_c$$

for local functions S_c on $\mathbb{R}\times M$. We generalize this by looking for a function G on E which pulls back by σ_c to each particular solution S_c, that is,

$$S_c := \sigma_c^* G.$$

It is sufficient to consider $\mathrm{d}_D G$ because θ_L has no $\mathrm{d}h^a$ components in local coordinates $(\bar t,\bar x^a,h^a)$ on E obtained from (t,x^a,u^a) by locally inverting the expressions for the h^a in terms of the u^a.

Conversely, it was shown in [11] that complete integrability (that is, the involutivity of h_a) implies the existence of a solution to the equation $\mathrm{d}_D G=\theta_L$ so that the local functions S_c guarantee the existence of a local function G on E such that $\sigma_c^*G=S_c$.

In the coordinates $(\bar t,\bar x^a,h^a)$ the Hamilton–Jacobi equation on E becomes

$$L\,\mathrm{d}t+\frac{\partial L}{\partial u^a}(\mathrm{d}\bar x^a-u^a\,\mathrm{d}\bar t)=\Gamma(G)\,\mathrm{d}\bar t+\frac{\partial G}{\partial \bar x^a}(\mathrm{d}\bar x^a-u^a\,\mathrm{d}\bar t)$$

where the u^a are of course functions of t,x^a and the values of the h^a on each leaf. In the $(\bar t,\bar x^a,h^a)$ coordinates we have

$$\Gamma=\frac{\partial}{\partial\bar t}+u^a\frac{\partial}{\partial\bar x^a}$$

and the equations

$$\Gamma(G)=L \qquad \frac{\partial G}{\partial\bar x^a}=\frac{\partial L}{\partial u^a}$$

give the previous Hamilton–Jacobi equations (on $\mathbb{R}\times M$) when pulled back by any one of the sections σ_c corresponding to the leaves.

Thus finding a solution (D,G) means finding a complete solution of the Hamilton–Jacobi equation in the traditional sense. In this way we achieve our goal of connecting the traditional idea of a complete solution of the Hamilton–Jacobi equation with a solution of an equation on the leaves of a foliation of E.

As indicated above the proof given in [11] of the Liouville–Arnol'd theorem shows that complete integrability guarantees a complete solution of the Hamilton–Jacobi equation. We will now show that the converse is also true.

Proposition 3.2. *The existence of a complete solution of the Hamilton–Jacobi equation on E guarantees complete integrability.*

Proof. Let G be a complete solution of the Hamilton–Jacobi equation, so that $\mathrm{d}_D G=\theta_L$ and hence $\mathrm{d}_D\theta_L=0$. We need to show that $\sigma_c^*\,\mathrm{d}\theta_L=0$. However, since the restriction of D to E_c is just the tangent bundle of the image of σ_c in E, $\sigma_c^*\,\mathrm{d}\theta_L=\sigma_c^*\,\mathrm{d}_D\theta_L=0$. □

From now on we will assume that our system is completely integrable with respect to L and that the n first integrals f^a are in involution.

3.2. $\mathbb{R} \times TM$ separability

It will be useful to adopt a notation for declaring functional dependency. Let K be a function on E with local coordinates (t, x^a, u^a) then $K[t, u^a]$ will be used in place of K to indicate that the coordinate representation of K does not depend on the functions x^a. It will also be useful to have some generic coordinates (z^0, z^a, f^a) for E adapted to the level sets of the f^a. In general, the maps z^0 and z^a will not simply be composites of the maps t, x^a but will also involve the f^a (and hence the u^a).

Definition 3.3 (Separability). *A complete solution G of the Hamilton–Jacobi equation is* separable *with respect to coordinates (z^0, z^a, f^a) if there exist non-zero functions $G_0 \ldots G_n$ such that*

$$G[z^0, z^a, f^a] = G_0[z^0, f^a] + \cdots + G_n[z^n, f^a].$$

A complete solution G of the Hamilton–Jacobi equation is fully separable *with respect to coordinates (z^0, z^a, f^a) if it is separable and*

$$G[z^0, z^a, f^a] = G_0[z^0, f^a] - f^b z^b.$$

(We use the summation convention on terms like $f^b z^b$ even though the indices are irregularly placed.)

Remark. The idea behind full separability is to mimic the best possible situation with conventional separability where there are n ignorable coordinates. In that case the coefficients of f^b are just the ignorable coordinates themselves. The following lemma shows that something quite startling happens when we try full separability of this type and indicates that when fully separable coordinates can be found the second set of involutive integrals from the Liouville–Arnol'd theorem appear as the natural extension of ignorable coordinates (see also corollary 3.6 following the main theorem and section 3.3). It illustrates the simplicity of the tangent bundle description of the Hamilton–Jacobi equation.

Lemma 3.4. *If the Hamilton–Jacobi equation admits a solution of the form*

$$G[z^0, z^a, f^a] = \bar{G}_0[z^0, f^a] - f^b z^b$$

then it also admits a solution of the form

$$G[z^0, y^a, f^a] = G_0[z^0, f^a] - f^b y^b$$

in coordinates (z^0, y^a, f^a) where $\Gamma(G_0) = L$, and $\Gamma(y^a) = 0$.

Proof. We assume that there exists a solution to the Hamilton–Jacobi equation of the form $G[z^0, z^a, f^a] = \bar{G}_0[z^0, f^a] - f^b z^b$. We define (y^a) by $y^a := z^a - (\partial\bar{G}_0[z^0, f^b])/(\partial f^a)$, and now from $\mathrm{d}y^a = -\mathrm{d}\big((\partial\bar{G}_0[z^0, f^b])/(\partial f^a)\big) + \mathrm{d}z^a$ we see that $\mathrm{d}y^a \wedge \mathrm{d}f^b \neq 0$, and $\mathrm{d}y^a \wedge \mathrm{d}z^0 \neq 0$, so that (z^0, y^a, f^a) are independent coordinates. Changing coordinates from (z^0, z^a, f^a) to (z^0, y^a, f^a) allows us to define G_0 so that

$$\begin{aligned} G[z^0, y^a, f^a] &= \bar{G}_0[z^0, f^a] - f^b\left[y^b + \frac{\partial\bar{G}_0[z^0, f^a]}{\partial f^b}\right] \\ &= G_0[z^0, f^a] - f^b y^b. \end{aligned}$$

It remains to show that $\Gamma(G_0) = L$ and $\Gamma(y^a) = 0$. Now $\mathrm{d}_D G = \theta_L$ by assumption so, in coordinates (z^0, y^a, f^a),

$$\begin{aligned} \mathrm{d}G &= \theta_L + \frac{\partial G}{\partial f^a}\,\mathrm{d}f^a \\ \Rightarrow \mathrm{d}\theta_L &= -\mathrm{d}\left(\frac{\partial G}{\partial f^a}\,\mathrm{d}f^a\right) \\ &= -\mathrm{d}\left(\frac{\partial G}{\partial f^a}\right) \wedge \mathrm{d}f^a. \end{aligned}$$

Because $\mathrm{d}\theta_L$ has maximal rank

$$\mathrm{d}\left(\frac{\partial G}{\partial f^1}\right) \wedge \ldots \wedge \mathrm{d}\left(\frac{\partial G}{\partial f^n}\right) \wedge \ldots \wedge \mathrm{d}f^1 \wedge \ldots \wedge \mathrm{d}f^n \neq 0$$

implying the linear independence of these $2n$ forms. Hence $\Gamma \lrcorner\, \mathrm{d}\theta_L = 0$ and $\Gamma(f^a) = 0$ imply $\Gamma((\partial G[z^0, z^b, f^b])/(\partial f^a)) = 0$ and we have $\Gamma(z^a) = (\bar{G}_0[z^0, f^b])/(\partial f^a)$ so that $\Gamma(y^a) = 0$. $\Gamma(G) = L$ then gives $\Gamma(G_0) = L$ as required. □

The dt-free Cartan symmetries corresponding to a set of involutive first integrals commute amongst themselves and with Γ, moreover they (along with Γ) form a basis for tangent spaces of the common level sets of the first integrals. Consequently, these $n + 1$ fields are coordinate fields for the level sets. The obvious question then is whether or not these coordinates provide separability for the corresponding Hamilton–Jacobi equation.

The answer is generally no (the two-dimensional Kepler problem is easily shown to be a counter-example). The same observations can be made for the θ_L-free Cartan symmetries and also they are all inappropriate for the task (θ_L annihilates these fields). However, as it happens the appropriate coordinate fields, when they exist, are elements of the transverse fields of Cartan symmetries of the involutive integrals.

For the sake of brevity we will write $X_a := \Theta_L^{-1}(\mathrm{d}f^a)$ for each of our involutive integrals f^a. We will write Y_a for the dt-free elements of $[X_a]$ described in proposition 2.3. In the following theorem we look for $W_a \in [X_a]$ with $W_a = Y_a + \alpha_a\Gamma$ so that, along with Γ, these fields provide coordinate fields for coordinates (y^0, y^a) on the level sets of the f^a in which the solution of the Hamilton–Jacobi equation separates.

Theorem 3.5 (Hamilton–Jacobi separability). *The Hamilton–Jacobi equation admits a solution of the form*

$$G[z^0, z^a, f^a] = \bar{G}_0[z^0, f^a] - f^b z^b$$

if and only if

$$\Gamma(\alpha_a) = 0$$

where the α_a are defined by $\alpha_a L := -f^a - \theta_L(Y_a)$.

Proof. From the preceding lemma, we need only to prove that these conditions imply, and are implied by, the existence of a solution to the Hamilton–Jacobi equation of the form: $G[y^0, y^a, f^a] = G_0[y^0, f^a] - f^b y^b$, where the coordinates satisfy the relations $\Gamma(G_0) = L$, $\Gamma(y^a) = \Gamma(f^a) = 0$.

Proof of necessity. Assume a separable solution $G[y^0, y^a, f^a] = G_0[y^0, f^a] - f^b y^b$ with $\Gamma(G_0) = L$ and $\Gamma(y^a) = 0$. Now in these coordinates we know that $\Gamma = \Gamma(y^0)\partial/(\partial y^0) + \Gamma(y^a)\partial/(\partial y^a)$ but $\Gamma(y^a) = 0$ so without loss of generality we can write $\Gamma(y^0) = 1$.

We wish to show that $\frac{\partial}{\partial y^a} \in [Y_a]$, that is, $\frac{\partial}{\partial y^a} - Y_a = \alpha_a \Gamma$, with α_a satisfying the required conditions.

We start with the Hamilton–Jacobi equation $\theta_L = \mathrm{d}_D G$, and substitute in our separated expression for G, giving us

$$\begin{aligned}\theta_L &= \mathrm{d}_D G_0 - f^a\,\mathrm{d}y^a \\ &= \Gamma(G_0)\mathrm{d}y^0 - f^a\,\mathrm{d}y^a.\end{aligned}$$

Now taking the Lie derivative of this with respect to the vector fields $\partial/(\partial y^b)$, gives:

$$\begin{aligned}&\mathcal{L}_{\partial/(\partial y^b)}\theta_L = 0 \\ \Rightarrow\ &\mathcal{L}_{\partial}/(\partial y^b)\,\mathrm{d}\theta_L = 0\end{aligned}$$

so that $\partial/(\partial y^b)$ is a Cartan symmetry.

In addition,

$$\begin{aligned}\frac{\partial}{\partial y^b} \lrcorner\, \mathrm{d}\theta_L &= \mathcal{L}_{\partial/(\partial y^b)}\theta_L - \mathrm{d}\left(\theta_L\left(\frac{\partial}{\partial y^b}\right)\right) \\ &= 0 + \mathrm{d} f^b\end{aligned}$$

so that $\frac{\partial}{\partial y^b} \in [Y_a]$.

Evaluating θ_L on $\frac{\partial}{\partial y^a} - Y_a = \alpha_a \Gamma$ gives $-f^a - \theta_L(Y_a) = \alpha_a L$ as required.

Next we prove $\Gamma(\alpha_a) = 0$. To accomplish this we take the Lie derivative of the relation $\frac{\partial}{\partial y^a} - Y_a = \alpha_a \Gamma$ by Γ, written as $\frac{\partial}{\partial y^0}$, giving

$$\left[\frac{\partial}{\partial y^0}, \partial/(\partial y^a)\right] - [\Gamma, Y_a] = \Gamma(\alpha_a)\Gamma.$$

Since $\left[\frac{\partial}{\partial y^0}, \frac{\partial}{\partial y^a}\right] = 0$ by assumption and $[\Gamma, Y_a] = 0$ by definition of the Y_a, we obtain $\Gamma(\alpha_a) = 0$.

Proof of sufficiency. Starting with the vector fields Y_a, Γ, we introduce new fields W_a, Γ, where $W_a \in [Y_a]$ defined by $W_a := Y_a + \alpha_a \Gamma$, and show that the conditions on the α_a's imply that Γ, W_a are coordinate fields of the coordinates of a fully separable solution to the Hamilton–Jacobi equation.

We first require that there exist coordinates (y^0, y^a) forming a chart with the f^a such that $\Gamma = \frac{\partial}{\partial y^0}$ and $W_a = \frac{\partial}{\partial y^a}$. Hence, we require $[W_a, \Gamma] = 0$, and $[W_a, W_b] = 0$. Substituting the relation $W_a = Y_a + \alpha_a \Gamma$ into these, we find

$$\begin{aligned}[W_a, \Gamma] &= [Y_a + \alpha_a\Gamma, \Gamma] \\ &= [Y_a, \Gamma] + [\alpha_a\Gamma, \Gamma] \\ &= 0\end{aligned}$$

as $\Gamma(\alpha_a) = 0$. Doing the same for the second condition is a little more complex,

$$[W_a, W_b] = [Y_a, Y_b] + Y_a(\alpha_b)\Gamma - Y_b(\alpha_a)\Gamma + \alpha_a\Gamma(\alpha_b)\Gamma - \alpha_b\Gamma(\alpha_a)\Gamma$$

and since $[Y_a, Y_b] = 0$ and $\Gamma(\alpha_a) = 0$ by assumption, we obtain

$$[W_a, W_b] = (Y_a(\alpha_b) - Y_b(\alpha_a))\Gamma.$$

To show that $Y_a(\alpha_b) - Y_b(\alpha_a) = 0$ we use all the assumptions to obtain $Y_a(\alpha_b L) = Y_b(\alpha_a L)$ (this is straightforward). Then use the Leibniz rule on this result to obtain

$$Y_b(\alpha_a)L - Y_a(\alpha_b)L = \alpha_b Y_a(L) - \alpha_a Y_b(L).$$

Finally, use $Y_a(L) = -\alpha_a\Gamma(L)$ to show that the right-hand side of the above equation is zero. Hence $[W_a, W_b] = 0$.

Starting with $W_a := Y_a + \alpha_a\Gamma$, and substituting in the definition of α_a, we obtain

$$W_a = Y_a + \left(\frac{-f^a - \theta_L(Y_a)}{L}\right)\Gamma.$$

Evaluating θ_L on this and using $\theta_L(\Gamma) = L$ gives

$$\theta_L(W_a) = -f^a.$$

As this is true for $a = 1, \ldots, n$, we have

$$\theta_L = \theta_0[y^0, y^a, f^a]\,\mathrm{d}y^0 - f^b\,\mathrm{d}y^b.$$

Now from $\theta_L = \mathrm{d}_D G$, we obtain $\frac{\partial G}{\partial y^a} = -f^a$, which implies that G takes the form $G = G_0[y^0, f^a] - f^b y^b$ as required. □

Corollary 3.6. *If the Hamilton–Jacobi equation has a solution of the form*

$$G[y^0, y^a, f^a] = G_0[y^0, f^a] - f^b y^b$$

then there exists an equivalent solution of the form

$$\hat{G}[y^0, y^a, f^a] = \hat{G}_0[y^0] - f^b y^b.$$

Proof. A simple modification of the proof of lemma 3.4 gives $\Gamma\left(\frac{\partial G}{\partial f^a}\right) = 0$ in coordinates (y^0, y^a, f^a). Substituting in our expression for G, we have $\Gamma\left(\frac{\partial G_0}{\partial f^a}\right) = 0$ since $\Gamma(y^a) = 0$. This gives $(\partial G_0[y^0, f^b])/(\partial f^a) = H[f^b]$. Hence

$$G[y^0, y^a, f^a] = \hat{G}_0[y^0] + \bar{H}[f^a] - f^b y^b.$$

The Hamilton–Jacobi equation $\mathrm{d}_D G = \theta_L$ indicates that the term $\bar{H}[f^a]$ is irrelevant so that

$$\hat{G}[y^0, y^a, f^a] := \hat{G}_0[y^0] - f^b y^b$$

is a solution as required. □

Remark.

(a) The computational significance of our separability theorem is this: suppose that the conditions of the theorem apply, then the y^a can be very simply calculated from the W_a and Γ. These $n + 1$ vector fields commute and so on the $(n + 1)$-dimensional level sets E_c the $\mathrm{d}_D y^a$ can be constructed algebraically and hence the $\mathrm{d}y^a$ are known modulo the $\mathrm{d}f^a$. Finally, we integrate these 1-forms to obtain the y^a. A by-product is an explicit representation of G, but this is hardly important because we have the remaining first integrals. This computation is particularly important when the original first integrals arise through the Noether–Cartan theorem from n commuting symmetries of $\mathrm{d}\theta_L$ rather than the Hamilton–Jacobi equation. See our third example for an explicit calculation.

(b) We pass on a significant comment made to us by Willy Sarlet. The proof of the main theorem obscures the fact that α_a are chosen so that the element $W_a := Y_a + \alpha_a\Gamma$ of $[X_a]$ satisfies $\mathcal{L}_{W_a}\theta_L = 0$. (This follows from $\alpha_a L = -f^a - \theta_L(Y_a)$ without the further assumption that $\Gamma(\alpha_a) = 0$.) We then observe that the $[X_a, X_b] = 0$ implies that $[Y_a, Y_b] = 0$ (see proposition 2.3) but that $[W_a, W_b] \neq 0$ unless we assume $\Gamma(\alpha_a) = 0$. This is a curious feature of the transverse fields, $[X_a]$ of completely integrable systems.

(c) It may appear that the conditions of the main theorem are so strong as to be useless because by satisfying them we straighten out the flow of Γ. We remind the reader that the initial assumption of complete integrability amounts to the same thing because of the Liouville–Arnol'd theorem. All we are doing is requiring that the Hamilton–Jacobi equation have an additional structure which makes the construction of the remaining integrals independent of the function G. We also remark that the condition $\Gamma(\alpha^a) = 0$ is not terribly strong in practice: in our examples the α^a are usually constants and they certainly do not have to satisfy any relation with the f^a's or the complementary set of integrals.

The separability theorem takes on a simple form when the Lagrangian is itself a first integral, in particular when it is a composite of the involutive integrals. The following corollary shows that the Lagrangian is always a composite of the involutive integrals associated with a fully separable solution. The proof is short and straightforward and can be found in [10].

Corollary 3.7 (Geodesic motion). *Let Γ be the Euler field for geodesic motion on an n-dimensional (pseudo-) Riemannian manifold (M, g) with Lagrangian $L := g_{ab}u^a u^b$. If $\{f^1, \dots, f^n\}$ are an involutive set with respect to L then the Hamilton–Jacobi equation is fully separable with respect to this set if and only if L is a composite of the f^a's.*

This result has broader applicability: suppose that L is one of the f^a's and that we pose the question of separability with respect to the complementary involutive integrals, the $\frac{\partial G}{\partial f^a}$'s (in the coordinates (y^0, y^a, f^a) for example). Because L is certainly not a composite of these complementary integrals the corollary tells us that the system is not separable with respect to these involutive integrals. The principle here is that full separability is in general a function of the involutive integrals appearing in a given complete solution of the Hamilton–Jacobi equation. For this reason it is rather difficult to formulate results claiming full separability or its absence based on the equations of motion alone.

3.3. $\mathbb{R} \times M$ *separability*

Now we turn to the issue of full separability in coordinates for $\mathbb{R} \times M$. In this section we will need local coordinates (x^0, x^a) for $\mathbb{R} \times M$ which do not in general respect its product structure. The associated coordinates for $\mathbb{R} \times TM$ are $(\bar{x}^0, \bar{x}^a, f^a)$ where $\bar{x}^0(t, x, u) := x^0(t, x)$, $\bar{x}^a(t, x, u) := x^a(t, x)$.

It should be clear from the earlier discussion that a fully separated solution

$$G[\bar{x}^0, \bar{x}^a, f^a] = G_0[\bar{x}^0, f^a] - f^b \bar{x}^b$$

on E produces fully separated solutions $S_c := \sigma_c^* G$ because $\sigma_c^* \bar{x}^\alpha = x^\alpha$ and conversely, the existence of these separated solutions S_c guarantees the existence of a separated solution G such that $\sigma_c^* G = S_c$. The contrapositive of lemma 3.4, with (z^0, z^a, f^a) replaced by $(\bar{x}^0, \bar{x}^a, f^a)$, excludes anything other than isolated fully separated solutions on $\mathbb{R} \times M$ if $\Gamma(\alpha_a) \neq 0$.

There remains the issue of deciding when 'upstairs' separability implies 'downstairs' separability. As we will show this depends on the projectability under $\tau : E \to \mathbb{R} \times M$ of the $[W_a]$ of theorem 3.5. (A vector field Z on E is *projectable* if there exists a vector field X on $\mathbb{R} \times M$ such that $X_{(t,x)} = \tau_{*(t,x,u)} Z_{(t,x,u)}$ for all $(t, x, u) \in E$ and we write $X = \tau_* Z$.)

Theorem 3.8. *If*

$$G[\bar{x}^0, \bar{x}^a, f^a] = G_0[\bar{x}^0, f^a] - f^b \bar{x}^b$$

is a solution of the Hamilton–Jacobi equation, then $\frac{\partial}{\partial \bar{x}^a}$ is a projectable Cartan symmetry belonging to f^a and $\frac{\partial f^b}{\partial x^a} = 0$.

Proof. Use $\mathrm{d}(\mathrm{d}_D G) = \mathrm{d}\theta_L$ to obtain

$$\mathrm{d}\theta_L = \mathrm{d}\left(\frac{\partial G_0}{\partial \bar{x}^0}\right) \wedge \mathrm{d}\bar{x}^0 - \mathrm{d}f^b \wedge \mathrm{d}\bar{x}^b$$

so that $\mathcal{L}_{\partial/\partial\bar{x}^a}\,\mathrm{d}\theta_L = 0$. The projectability of the $\frac{\partial}{\partial\bar{x}^a}$ can be seen by writing them as linear combinations of the coordinate fields for the coordinates (x^0, x^a, u^a) using

$$\frac{\partial}{\partial x^a} = \frac{\partial}{\partial \bar{x}^a} + \frac{\partial f^b}{\partial x^a}\frac{\partial}{\partial f^b} \qquad \text{and} \qquad \frac{\partial}{\partial u^a} = \frac{\partial f^b}{\partial u^a}\frac{\partial}{\partial f^b}.$$

To see that $\frac{\partial f^b}{\partial x^a} = 0$, we use the fact that symmetries of $\mathrm{d}\theta_L$ are symmetries of Γ. The fact that $\frac{\partial}{\partial x^a} = \tau_* \frac{\partial}{\partial \bar{x}^a}$ then forces $\frac{\partial}{\partial x^a} = \frac{\partial}{\partial \bar{x}^a}$. □

Remark. It is a simple consequence of this theorem that the coordinates x^a of theorem 3.8 are ignorable, that is, $\frac{\partial L}{\partial x^a} = 0$.

Next we give a loose converse to this theorem, but first we remark that each transverse field belonging to Γ contains at most one projectable element and that projectable elements belonging to distinct transverse fields do not generally commute.

Theorem 3.9. *Let*

$$G[y^0, y^a, f^a] = G_0[y^0, f^a] - f^b y^b$$

be a solution of the Hamilton–Jacobi equation with $\Gamma(y^0) = 1$ *and* $\Gamma(y^a) = 0$. *If each* $\left[\frac{\partial}{\partial y^a}\right]$ *contains a projectable element and these* n *elements commute pairwise then* $S_c := \sigma_c^* G$ *is a fully separable solution of* $\mathrm{d}S = \sigma^*\theta_L$ *for each* c *on some open, connected subset of* $\mathbb{R}^n$.

Proof. Let $Z_1, \ldots, Z_n$ be the pairwise commuting, projectable (and necessarily independent) fields and denote their projections by $\tau_* Z_a$. It is straightforward to verify that we can introduce local coordinates (x^0, x^n) for $\mathbb{R} \times M$ so that $\tau_* Z^a$ are the coordinate fields corresponding to the x^a. In the usual induced local coordinates $(\bar{x}^0, \bar{x}^a, f^a)$ for E the fields $\frac{\partial}{\partial\bar{x}^a}$ project to the $\tau_* Z_a = \frac{\partial}{\partial x^a}$ (see the proof of theorem 3.8). Hence $Z_a - \frac{\partial}{\partial\bar{x}^a}$ is vertical for each a. Since, for each a, both of these fields are tangent to the images $\sigma_c(\mathbb{R} \times M)$ for all c in some open, connected subset $\mathcal{D}$ of $\mathbb{R}^n$, they must coincide.

Now we choose $(y^0, \bar{x}^a, f^a)$ as local coordinates for E and we see that

$$y^b = \bar{x}^b + H^b[y^0, f^a]$$

for some functions H^b. Applying this to the expression for our fully separated solution G we obtain

$$G[y^0, \bar{x}^a, f^a] = \hat{G}_0[y^0, f^a] - f^b \bar{x}^b$$

and so

$$\sigma_c^* G = \hat{G}_0[\sigma_c^* y^0, c^a] - c^b x^b$$

for each $c \in \mathcal{D}$. It only remains to show that $\mathrm{d}(\sigma_c^* y^0) \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \neq 0$ so that $(\sigma_c^* y^0, x^a)$ can be used as local coordinates for $\mathbb{R} \times M$. This is true because $\sigma_c^*\left(\mathrm{d}y^0 \wedge \mathrm{d}\bar{x}^1 \wedge \ldots \wedge \mathrm{d}\bar{x}^n\right) = \mathrm{d}(\sigma_c^* y^0) \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ and because the pullback is one to one and onto. Since G is a solution of $\mathrm{d}_D G = \theta_L$, $S_c := \sigma_c^* G$ is a solution of the usual Hamilton–Jacobi equation for each $c \in \mathcal{D}$. □

Remark. This last result is a little unsatisfactory because it involves a different base coordinate, $\sigma_c^* y^0$, for each c. Ideally one would like to show that $\sigma_k{}^* y^0$ depends only on $\sigma_c^* y^0$ and k for some fixed $c \in \mathcal{D}$. However, the overall result of theorems 3.8 and 3.9 is that full separability on the base occurs if and only if the conditions of theorem 3.5 hold and there are n commuting Noether symmetries of $\mathrm{d}\theta_L$ corresponding to n ignorable coordinates.

4. The Hamilton–Jacobi equation for contact flows

In general, a system of second-order differential equations will not be Euler–Lagrange. However, it was shown in [11] that, at least locally, there exists a 2-form which serves exactly the same purpose as $\mathrm{d}\theta_L$ in the Euler–Lagrange case. This is because such equations guarantee the local existence of a *contact form* on E. We will give a brief précis of definitions and results from [11].

A *contact form* on a $(2n + 1)$-dimensional manifold P is a smooth 1-form ω such that $\omega \wedge (\mathrm{d}\omega)^n$ is nowhere zero. A contact form uniquely determines a vector field V, called the *Reeb field*, on the manifold such that $\omega(V) = 1$ and $\mathcal{L}_V\omega = 0$. It is apparent that an equivalent pair of conditions is $\omega(V) = 1$ and $V \in \ker \mathrm{d}\omega$. The flow of V is called the *contact flow* and a manifold which admits a contact form is called a *contact manifold*.

It should be clear that any regular Lagrangian system provides a contact form for E by virtue of the properties of $\mathrm{d}\theta_L$, namely $\omega := \theta_L - \mathrm{d}G + \mathrm{d}t$ where G satisfies $\Gamma(G) = L$. The corresponding Reeb field is the Euler field Γ.

It is shown in [11] that *every* second-order equation field Γ has a contact form, called a Γ-*contact form*, associated with it. For example, if $h^1, \ldots, h^{2n}$ are functionally independent first integrals on some domain then $\omega := h^1\,\mathrm{d}h^2 + \cdots + h^{2n-1}\,\mathrm{d}h^{2n} + \mathrm{d}t$ is such a form. Explicit construction of such forms cases usually requires some knowledge of first integrals (as in the inverse problem in the calculus of variations).

Suppose that we have a (possibly local) Γ-contact form ω on E for a second-order differential equation field Γ. It is shown in [11] that the 2-form $\mathrm{d}\omega$ replaces $\mathrm{d}\theta_L$ in the definitions of Cartan fields, involutiveness and complete integrabilty. Proposition 2.2 holds with $\mathrm{d}f = X_f \lrcorner\, \mathrm{d}\omega$ and $\omega(X_f) = 0$ and similarly for X_g. Proposition 2.3 also holds and we will use Y_f to identify this element of $[X_f]$. Indeed, the entire Liouville–Arnol'd theorem is true for Γ-contact flows. The generalized Hamilton–Jacobi equation on E is

$$\mathrm{d}_D G = \omega$$

where d_D is as before (and a solution on $\mathbb{R} \times M$ satisfies $\sigma^*\omega = \mathrm{d}S$). It is a simple matter to show that proposition 3.2 continues to hold so we will assume that this flow is completely integrable and use involutive first integrals f^a. In the coordinates $(\bar{t}, \bar{x}^a, f^a)$ this generalized equation has components

$$\Gamma(G) = 1 \qquad \text{and} \qquad \frac{\partial G}{\partial \bar{x}^a} = \omega\left(\frac{\partial}{\partial \bar{x}^a}\right).$$

We will stick with our definitions from the previous sections of separable and fully separable solutions. Lemma 3.4 stands with $\Gamma(G_0) = 1$ replacing $\Gamma(G_0) = L$. Theorem 3.5 becomes (with $Y_a \in [X_a]$ as explained above):

Theorem 4.1. *The generalized Hamilton–Jacobi equation always admits a solution of the form*

$$G[z^0, z^a, f^a] = \bar{G}_0[z^0, f^a] - f^b z^b.$$

Proof. The conditions of the previous theorem become

$$\Gamma(\alpha_a) = 0$$

where $\alpha_a := -f^a - \omega(Y_a)$. It is a simple matter to show that $\Gamma(\alpha_a) = 0$ is identically satisfied for a completely integrable system. The proof of the result then proceeds as before: from the generalization of lemma 3.4, we need only to prove that the existence of a solution to the generalized Hamilton–Jacobi equation of the form $G[y^0, y^a, f^a] = G_0[y^0, f^a] - f^b y^b$, where the coordinates satisfy the relations $\Gamma(y^0) = 1$, $\Gamma(y^a) = 0$ is equivalent to $\Gamma(\alpha_a) = 0$.

The proof of necessity is a simple modification of the one in the previous section. The proof of sufficiency is simpler than before because the condition $\theta_L(\Gamma) = L$ becomes $\omega(\Gamma) = 1$. The remainder of the proof is as before. □

This result means that we can construct the remaining first integrals directly from the symmetries themselves. See example 5.3 below. It also has implications for the Euler–Lagrange equations: we are free to modify the Cartan 1-form θ_L by the addition of exact 1-forms, this is a type of gauge freedom. In particular, we can set

$$\omega := \theta_L - \mathrm{d}G + \mathrm{d}t$$

(and so $\mathrm{d}\omega = \mathrm{d}\theta_L$). But this ω is a Γ-contact form, so in this gauge the Hamilton–Jacobi equation is fully separable if the Euler–Lagrange system is completely integrable. Moreover, in this gauge it is a simple matter to show $\alpha_a := -f^a - \omega(Y_a) = -(\partial G)/(\partial y^a)$ where y^a are the separable coordinates of theorem 4.1: these are exactly the remaining first integrals given by the Liouville–Arnol'd theorem!

5. Examples

Example 5.1 (The Toda lattice). *The Toda lattice [8, 17] is a completely integrable system that is not conventionally separable. The configuration space is $\mathbb{R}^3$ and the Lagrangian is*

$$L := \tfrac{1}{2}({u^1}^2 + {u^2}^2 + {u^3}^2) - \left(\mathrm{e}^{(x^1-x^2)} + \mathrm{e}^{(x^2-x^3)} + \mathrm{e}^{(x^3-x^1)}\right).$$

We know three involutive first integrals with respect to this Lagrangian:

$$\begin{aligned} f^1 &= u^1 + u^2 + u^3 \\ f^2 &= u^1u^2u^3 - u^1\mathrm{e}^{(x^2-x^3)} - u^2\mathrm{e}^{(x^3-x^1)} - u^3\mathrm{e}^{(x^1-x^2)} \\ f^3 &= \tfrac{1}{2}({u^1}^2 + {u^2}^2 + {u^3}^2) + \mathrm{e}^{(x^1-x^2)} + \mathrm{e}^{(x^2-x^3)} + \mathrm{e}^{(x^3-x^1)} \end{aligned}$$

the last of these being the Hamiltonian. We will show that these first integrals do not give us a fully separable solution to the Hamilton–Jacobi equation. Using the REDUCE [7] exterior calculus package EXCALC [15], we found the following dt*-free Cartan symmetries related to these first integrals:*

$$\begin{aligned} Y_1 &= -\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^3} \\ Y_2 &= \left(\mathrm{e}^{(x^2-x^3)} - u^2u^3\right)\frac{\partial}{\partial x^1} + \left(\mathrm{e}^{(x^3-x^1)} - u^1u^3\right)\frac{\partial}{\partial x^2} + \left(\mathrm{e}^{(x^1-x^2)} - u^1u^2\right)\frac{\partial}{\partial x^3} \\ &\quad + \left(u^2\mathrm{e}^{(x^3-x^1)} - u^3\mathrm{e}^{(x^1-x^2)}\right)\frac{\partial}{\partial u^1} + \left(u^3\mathrm{e}^{(x^1-x^2)} - u^1\mathrm{e}^{(x^2-x^3)}\right)\frac{\partial}{\partial u^2} \\ &\quad + \left(u^1\mathrm{e}^{(x^2-x^3)} - u^2\mathrm{e}^{(x^3-x^1)}\right)\frac{\partial}{\partial u^3} \\ Y_3 &= -\mathrm{e}^{(x^1+x^2+x^3)}\left(u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2} + u^3\frac{\partial}{\partial x^3}\right) + \left(\mathrm{e}^{(2x^1)} - \mathrm{e}^{(x^2+x^3)}\right)\frac{\partial}{\partial u^1} \\ &\quad + \left(\mathrm{e}^{(2x^2)} - \mathrm{e}^{(x^1+x^3)}\right)\frac{\partial}{\partial u^2} + \left(\mathrm{e}^{(2x^3)} - \mathrm{e}^{(x^1+x^2)}\right)\frac{\partial}{\partial u^3}. \end{aligned}$$

From these symmetries and the first integrals, we used EXCALC to calculate the values of α_1*,* α_2*, and* α_3*:*

$$\alpha_1 = 0$$
$$\alpha_2 = \frac{e^{(2x^1+x^3)}u^3(u^1-u^2) + e^{(2x^2+x^1)}u^1(u^2-u^3) + e^{(2x^3+x^2)}u^2(u^3-u^1)}{e^{(2x^1+x^3)}(u^1-u^2) + e^{(2x^2+x^1)}(u^2-u^3) + e^{(2x^3+x^2)}(u^3-u^1)}$$
$$\alpha_3 = -1.$$

We find $\Gamma(\alpha_1) = 0$*, and* $\Gamma(\alpha_3) = 0$*, but* $\Gamma(\alpha_2) \neq 0$*. So these first integrals do not produce a fully separable coordinate system for the Hamilton–Jacobi equation at either level. Unfortunately, currently we do not know whether there exists a different set of involutive first integrals which would give us a separable solution.*

Example 5.2 (The Kerr metric). *This example is treated in full in [10], we restrict ourselves to a summary. The geodesic equations of the Kerr metric are completely integrable, the responsible involutive integrals are*

$$f^1 := \tfrac{1}{2}g_{ab}u^a u^b \qquad f^2 := -g_{0a}u^a$$
$$f^3 := -g_{3a}u^a \qquad f^4 := \tfrac{1}{2}K_{ab}u^a u^b.$$

f^1 *is just the geodesic Lagrangian* L *and* K *is Carter's Killing tensor [2]. The corresponding* dt*-free Cartan symmetries (see [12, 13]) are*

$$Y_1 = \frac{\partial}{\partial s} - \Gamma \qquad Y_2 = \frac{\partial}{\partial t}$$
$$Y_3 = \frac{\partial}{\partial \phi} \qquad Y_4 = -K^b_a u^a \frac{\partial}{\partial x^b} - \Gamma(K^b_a u^a)\frac{\partial}{\partial u^b}.$$

We can immediately apply corollary 3.7 and say that the system is separable with respect to this involutive set because $f^1 := L$*. It is a straightforward matter to show that the corresponding* α*'s are*

$$\alpha_1 = 1 \qquad \alpha_2 = 0 \qquad \alpha_3 = 0 \qquad \alpha_4 = \frac{f^4}{L}$$

(which are indeed all constant along the integral curves of Γ*) so that the separable coordinate fields for the level sets of the* f*'s are*

$$\Gamma \qquad W_1 := \frac{\partial}{\partial s} \qquad W_2 := \frac{\partial}{\partial t} \qquad W_3 := \frac{\partial}{\partial \phi} \qquad W_4 := Y_4 + \frac{f^4}{L}\Gamma$$

in the coordinate basis for (s, x^a, u^a)*. It is clear that the associated coordinates are not adapted to* $\mathbb{R}\times TM$ *and full separability is not achievable with coordinates from* $\mathbb{R}\times M$ *because* $[W_4]$ *has no projectable element, although Carter [2] shows that ordinary separability is possible. This is an example where downstairs full separability is not guaranteed by* n *quadratic integrals. From the point of view of this paper the Kerr metric is important because it provides an example of an upstairs fully separable system with* $\Gamma(L) = 0$ *with no full separability downstairs which, in addition, has a hidden symmetry.*

Example 5.3 (A contact flow on $\mathbb{R}^2$**).** *Consider*

$$\ddot{x} = \dot{y} \qquad \ddot{y} = y.$$

This system is not Euler–Lagrange [6], however, it produces a contact flow: for example, for the Γ*-contact form*

$$\omega := (u - y)\theta^x + \tfrac{1}{2}x\theta^y + (v - x)\psi^x + (\tfrac{1}{2}y - u)\psi^y + dt$$

in coordinates (t, x, y, u, v) *for* $E := \mathbb{R} \times T\mathbb{R}^2$. *The forms* $\theta^x := \mathrm{d}x - u\,\mathrm{d}t$, $\theta^y := \mathrm{d}y - v\,\mathrm{d}t$, *modified force forms* $\psi^x := -\frac{1}{2}(\mathrm{d}y + v\,\mathrm{d}t) + \mathrm{d}u$ *and* $\psi^y := \mathrm{d}v - y\,\mathrm{d}t$, *and* $\mathrm{d}t$, *form a convenient basis for* T^*E *(see [4]) because in this basis* $\mathrm{d}\omega$ *is free of* $\mathrm{d}t$ *terms:*

$$\mathrm{d}\omega = \theta^x \wedge \theta^y + 2\psi^x \wedge \theta^x - 2\psi^x \wedge \psi^y.$$

The system has two obvious commuting (point) symmetries generated by $\frac{\partial}{\partial t}$ *and* $\frac{\partial}{\partial x}$ *(regarded as vector fields on* E*).* $\mathcal{L}_{\partial/\partial t}\,\mathrm{d}\omega = 0 = \mathcal{L}_{\partial/\partial x}\,\mathrm{d}\omega$ *so that* $\partial/(\partial t) \lrcorner\, \mathrm{d}\omega$ *and* $\frac{\partial}{\partial x} \lrcorner\, \mathrm{d}\omega$ *produce involutary first integrals, namely* $f := u^2 + \frac{1}{2}(y^2 + v^2) - 2uy$ *and* $g := 2(y - u)$, *respectively. Using* $Y_f := \frac{\partial}{\partial t} - \Gamma$ *and* $Y_g := \frac{\partial}{\partial x}$ *a straightforward calculation shows that*

$$\alpha_f = 0 \qquad \alpha_g = -\tfrac{1}{2}g.$$

The generalized Hamilton–Jacobi equation is fully separable according to theorem 4.1 with coordinate fields

$$W_f := Y_f + \alpha_f \Gamma = -U\frac{\partial}{\partial \bar{x}} - V\frac{\partial}{\partial \bar{y}}$$

$$W_g := Y_g + \alpha_g \Gamma = -\frac{1}{2}g\frac{\partial}{\partial \bar{t}} + \left(1 - \tfrac{1}{2}gU\right)\frac{\partial}{\partial \bar{x}} - \frac{1}{2}gV\frac{\partial}{\partial \bar{y}}$$

$$\Gamma = \frac{\partial}{\partial \bar{t}} + U\frac{\partial}{\partial \bar{x}} + V\frac{\partial}{\partial \bar{y}}$$

in coordinates $(\bar{t}, \bar{x}, \bar{y}, f, g)$ *and where* U, V *are composites of these functions obtained by locally solving the expressions for* f, g *for* u, v, *respectively (in this way we hide any branches involved). Now we wish to find the corresponding coordinate functions, which we denote* r, s *(equivalently* $-\frac{\partial G}{\partial f}, -\frac{\partial G}{\partial g}$, *respectively), without finding* G. *Putting* $D := Sp\{\Gamma, W_f, W_g\}$ *we construct the following* 2*-form from* D^* *whose characteristic space in* D *is spanned by* Γ*:*

$$\Omega := (\mathrm{d}\bar{x} - U\,\mathrm{d}\bar{t}) \wedge (\mathrm{d}\bar{y} - V\,\mathrm{d}\bar{t}).$$

The 1*-forms*

$$\omega_g := \frac{W_f \lrcorner\, \Omega}{\Omega(W_f, W_g)} \qquad \textit{and} \qquad \omega_f := \frac{W_g \lrcorner\, \Omega}{\Omega(W_g, W_f)}$$

are both D*-closed and satisfy*

$$\omega_a(\Gamma) = 0 \qquad \omega_a(W_b) = \delta_{ab}.$$

(See [16] for details of this technique.)

It remains to integrate these two forms modulo f, g *to find* r, s.

The coordinate expressions are

$$\omega_g = \frac{V\,\mathrm{d}\bar{x} - U\,\mathrm{d}\bar{y}}{V} \qquad \omega_f = -\frac{\mathrm{d}\bar{y} - V\,\mathrm{d}\bar{t}}{V}.$$

Using the facts that $\mathrm{d}_D U = \mathrm{d}\bar{y}$ *and* $V\,\mathrm{d}_D V - \bar{y}\,\mathrm{d}\bar{y} = 0$ *we obtain*

$$r = t - \log(|y + v|) \qquad \textit{and} \qquad s = x - v - (u - y)\log(|y + v|)$$

in the original coordinates (t, x, y, f, g) *(up to arbitrary composites of* f, g*).*

This indicates that there is a separable solution (although not the one of theorem 4.1) corresponding to the ignorable coordinates t *and* x*:*

$$G[\bar{t}, \bar{x}, \bar{y}, f, g] = -\tfrac{1}{2}g\bar{x} + (1 - f)\bar{t} + F[\bar{y}, f, g]$$

where $F[\bar{y}, f, g]$ *is a tedious composite of* f, g *and* $\bar{y}$. *Notice that*

$$S_c[t, x, y] := \sigma_c^*(G[\bar{t}, \bar{x}, \bar{y}, f, g]) = -\tfrac{1}{2}c^1 x + (1 - c^2)t + H[y]$$

is the pullback of G associated with the section σ_c of $E \to \mathbb{R} \times M$ whose image is $E_c := \{v \in E : g(v) = c^1, f(v) = c^2\}$. We see in this case that full separability of the Hamilton–Jacobi equation occurs both on $\mathbb{R} \times M$ and on E. This example demonstrates that the applicability of the Hamilton–Jacobi method extends beyond Lagrangian/Hamiltonian systems. On the other hand, it can be regarded as an example where the Hamilton–Jacobi equation is never solved but merely serves as a construct by which four integrals are obtained from two symmetries.

6. Conclusion

It should be clear that in the case of completely integrable systems that separability of the Hamilton–Jacobi equation is quite straightforward (and satisfactory in as much as it is closely linked to the Noether–Cartan treatment of first integrals) when approached on E. Furthermore, it yields powerful results about whether a given set of involutive integrals are sufficient to guarantee full separability of the usual $\mathbb{R} \times M$ Hamilton–Jacobi equation.

We cannot yet relate our results to the well known ones on orthogonal separability and Killing tensors, and certainly Woodhouse's result on separability on one variable does not require anything so strong as complete integrability. On the positive side, we have generalized the Hamilton–Jacobi method to arbitrary systems of second-order ordinary differential equations and, along with paper on the Liouville–Arnol'd theorem, the current work once more shows how much useful structure can be transferred to the case of systems without a Lagrangian.

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